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Hermite Wavelet Based Method for the Numerical Solution of Linear and Nonlinear Delay Differential Equations

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Abstract

The theory and application of Delay Differential Equations (DDE's) have recently drawn a great deal of attention from Scientists, mathematicians and other disciplines. In this paper, Hermite Wavelet based Numerical Method for the solution of Delay Differential Equations (DDE's) is presented. The method is based on Hermite Polynomials. The present method reduces the computational difficulties of the other traditional methods and all the calculations can be made simple manipulations. Some illustrative numerical experiments are included to demonstrate the validity and applicability of the present technique. Comparison of numerical results explicitly reflects the high level of accuracy and reliability.

Keywords:

Hermite wavelet (HW); Delay Differential Equations (DDE's); Limit Point.

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1. Introduction

Delay differential equations (DDEs) arise in many areas of mathematical modeling, as physiological and pharmaceutical kinetics, chemical kinetics, population dynamics, the navigational control of ships and aircraft, infectious diseases, and more general control problems. Given the importance of applications of DDEs, many researchers have addressed this issue [1,2,3,10,11]. Since analytical solutions of the DDEs may be obtained only in very restricted cases; many methods have been proposed for numerical approximation of them. Moreover, there are a few works that analytic solutions are given. Especially, one of the main methods used on this area is finite difference method. Monotone iterative schemes for finite-difference solutions of reaction diffusion systems with time delay [17], the initial-value problem for linear delay partial differential equations of the parabolic type , Chebyshev polynomials method [19], Bellman's method of steps [7], Runge-Kutta method [7], spline methods [7,8], Radau IIA method [13], Multiquadric approximation scheme [18],variational iteration method (VIM) [15], Adomian decomposition method (ADM) [12] and homotopy perturbation method (HPM) .To study the behaviour of delay differential equations we need to find their exact solutions by the classical techniques. And then using these solutions one can know the behaviour of differential equation under the given different circumstances. We can not anticipate exact solution always. That is, suppose given delay differential equations are with complex coefficients, it becomes very difficult to find exact solution, therefore, we need numerical methods to solve such equations. In the present work we study the effect of delay on the problems for the different-order linear and nonlinear delay differential equations.

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Wavelets theory is a new and has been emerging tool in applied mathematical research area. Its applications have been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Moreover, wavelets establish a connection with fast numerical algorithms [18, 19]. Wavelets permit the accurate representation of a variety of functions and operaters. Wavelets are assumed as basis functions $\psi_{i,j}(x)$ in continuous time. Basis is a set of functions which are linearly independent and these linearly independent functions can be used to produce all admissible functions say $f(t)$. It is represented in wavelet space as $f(t)=\sum_{i,j} a_{ij}\psi_{ij}(x)$. Special feature of the wavelet basis is that all functions $\psi_{i,j}(x)$ are constructed from a single mother wavelet $\psi(x)$ which is a small pulse. Usually set of linearly independent functions (basis) created by translation and dilation of mother wavelet. In propesed method, unknown function appearing in the delay differential equations is replaced by series expansions of Hermite basis. After collocating this by suitable collocation points with given conditions then a system of linear or nonlinear equations is obtained which can be solved using iterative methods to give the unknown coefficients hence the solution is obtained by substituting these unknowns in unknown function. Let $(a,b)\subset R$ be an interval and $p(x):(a,b)\to R$ be continuous real valued functions. Throughout this paper we consider the delay equations given by

$$\frac{d^n y(x)}{dx^n}=f(x,y(x),y(g(x))),\ \ a<x<b. \tag{1}$$

subjected to following conditions: $y^i(0)=\alpha_i, i=0,1,2,...,N-1.$ where $\alpha_i, i=0,1,2,...$ are known real constants.

The paper is organized as follows. Section 2 gives some notations and basic definitions of the Hermite wavelets. Section 3 is devoted to the solution of DDEs using the HWM. In Section 4, we present some examples wherin their numerical results demonstrate the high accuracy and efficiency of the proposed method. The discussion and conclusion are presented in Section 5.

2. Preliminaries of Hermite Wavelets

Wavelets constitute a family of functions constructed from dialation and translation of a single function called mother wavelet. When the dialation parameter a and translation parameter b varies continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(x)=|a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right), \forall\ a,b\in R, a\neq 0.$$

If we restrict the parameters a and b to discrete values as $a=a_0^{-k}, b=nb_0a_0^{-k}, a_0>1, b_0>0.$ We have the following family of discrete wavelets

$$\psi_{k,n}(x)=|a|^{1/2}\,\psi(a_0^k x-nb_0), \forall a,b\in R, a\neq 0,$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$,then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as [17]

$$\psi_{k,n}(x)=\begin{cases}\dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}H_m(2^k x-2n+1), & \dfrac{n-1}{2^{k-1}}\le x<\dfrac{n}{2^{k-1}}\\ 0, & \text{otherwise}\end{cases} \tag{2}$$

where $m=0,1,\ldots,M-1$. Here $H_m(x)$ is Hermite polynomials of degree m with respect to weight function $W(x)=\sqrt{1-x^2}$ on the real line R and satisfies the following reccurence formula $H_0(x)=1$, $H_1(x)=2x$,

$$H_{m+2}(x)=2xH_{m+1}(x)-2(m+1)H_m(x) \tag{3}$$

where $m=0,1,2,\ldots$.

3. Hermite Wavelets Method of Solution

We would like to bring a solution function $y(x)$ under Hermite space by approximating the $y(x)$ by elements of Hermite wavelet basis as follows:

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x) \tag{4}$$

where $\psi_{n,m}(x)$ is given in Eq. (2).

We approximate $y(x)$ by truncating the series represented in Eq. 4 as,

$$y(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\psi(x) \tag{5}$$

where C and $\psi(x)$ are $2^{k-1}M \times 1$ matrix,

$$C^T = [C_{1,0},...C_{1,M-1},C_{2,0},...C_{2,M-1},...C_{2^{k-1},0},...C_{2^{k-1},M-1}] \tag{6}$$

$$\psi(x) = [\psi_{1,0},...\psi_{1,M-1},\psi_{2,0},...\psi_{2,M-1},...\psi_{2^{k-1},0},...\psi_{2^{k-1},M-1}] \tag{7}$$

Then $2^{k-1}M$ number of conditions required to determine $2^{k-1}M$ number of coefficients $C_{1,0},...C_{1,M-1},C_{2,0},...C_{2,M-1},...C_{2^{k-1},0},...C_{2^{k-1},M-1}$

Case 1. If given equation is of order one then there is a initial condition, namely $y(0) = \alpha_1$.

We see that there should be $2^{k-1}M - 1$ extra conditions to recover the unkown coefficients $C_{n,m}$. These conditions can be obtained by substituting Eq.5 in Eq. 1

$$\frac{d^n\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x)\right)}{dx^n} = f\left(x, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x), \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(g(x))\right), \tag{8}$$

Now we assume Eq. (8) is exact at $2^{k-1}M - 1$ limit points of the following sequence

$$x_i = \frac{1}{2}\left(1 + cos\frac{(i-1)\pi}{2^{k-1}M}\right), = 2,3,... \tag{9}$$

Then Eq. (8) will become

$$\frac{d^n\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i)\right)}{dx^n} = f\left(x_i, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x_i), \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(g(x_i))\right), \tag{10}$$

(10)

since one equation is furnished by the initial condition discussed in the section 1, remaining system of equations are obtained by Eq.10. So, totally we get $2^{k-1}M$ number of system of linear or nonlinear equations with $2^{k-1}M$ unkowns. by solving these equations we get $2^{k-1}M$ unknown coefficients values, substituting these unknown coefficients values in equation Eq.5 we get solution of equation Eq.(1)

Case 2. If given equation is of order two then there is a initial conditions, namely $y(0) = \alpha_1, y^{'}(0) = \alpha_2$.

We see that there should be $2^{k-1}M - 2$ extra conditions to recover the unkown coefficients $C_{n,m}$. These conditions can be obtained by substituting Eq.5 in Eq. 1

$$\frac{d^n\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x)\right)}{dx^n} = f\left(x, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x), \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(g(x))\right), \tag{11}$$

Now we assume Eq. (11) is exact at $2^{k-1}M - 2$ limit points of the following sequence

$$x_i = \frac{1}{2}(1+cos\frac{(i-1)\pi}{2^{k-1}M-1}), i = 2,3,... \quad (12)$$

Then Eq. (11) will become

$$\frac{d^n(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}C_{n,m}\psi_{n,m}(x_i))}{dx^n} = f(x_i, \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}C_{n,m}\psi_{n,m}(x_i), \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}C_{n,m}\psi_{n,m}(g(x_i)), \quad (13)$$

since two equations are furnished by the initial conditions discussed in the section 1, remaining system of equations are obtained by Eq.13. So, totally we get $2^{k-1}M$ number of system of linear or nonlinear equations with $2^{k-1}M$ unkowns. by solving these equations we get $2^{k-1}M$ unknown coefficients values, substituting these unknown coefficients values in equation Eq.5 we get solution of equation Eq.(1). Same procedure is repeated for higher order equations.

Theorem 1: let R^n is polynomial space of degree $n+1$ over field R and $y:[a,b]\to R^n$ be the solution of arbitrary second order differential equation then the solution for such differential equation by present method is exact.

4. Numerical Experiment

In this section, four experiments of DDEs are given and some comparisons are made to illustrate the efficiency of the method.

Test problem 4.1 Consider the following Delay Differential equation

$$y'(x) = \frac{1}{2}e^{\frac{x}{2}}y(\frac{x}{2}) + \frac{1}{2}y(x), \quad 0 < x < 1$$

Subject to the initial Condition $y(0)=1$.

Method of Implementation: Let us assume $y(x) = \sum_{j=0}^{M-1} C_i\psi_j$ for fixed k=1 and M=5

i.e.

$$y(x) = C_1\frac{2}{\sqrt{\pi}} + C_2\frac{2}{\sqrt{\pi}}[4x-2] + C_3\frac{2}{\sqrt{\pi}}[16(x^2-x)+2] + C_4\frac{2}{\sqrt{\pi}}[4x(3-4x)^2-2] + C_5\,64\,x(x-1)(1-2x)^2+2$$

Substituting these values of $y(x)\,\&\,y'(x)$ in the given equation, we get

$$C_2\frac{2}{\sqrt{\pi}}[4] + C_3\frac{2}{\sqrt{\pi}}[16(2x-1)] + C_4\frac{2}{\sqrt{\pi}}[192x^2-192x+36] + C_5\,64[16x^3-24x^2+10x-1] = \frac{1}{2}e^{\frac{x}{2}}$$

$$\left[C_1\frac{2}{\sqrt{\pi}} + C_2\frac{2}{\sqrt{\pi}}[4(\frac{x}{2})-2] + C_3\frac{2}{\sqrt{\pi}}[16((\frac{x}{2})^2-(\frac{x}{2}))+2] + C_4\frac{2}{\sqrt{\pi}}[4(\frac{x}{2})(3-4(\frac{x}{2}))^2-2] + C_5\,64\,(\frac{x}{2})((\frac{x}{2})-1)(1-2(\frac{x}{2}))^2+2\right]$$

$$+$$

$$\frac{1}{2}\left[C_1\frac{2}{\sqrt{\pi}} + C_2\frac{2}{\sqrt{\pi}}[4x-2] + C_3\frac{2}{\sqrt{\pi}}[16(x^2-x)+2] + C_4\frac{2}{\sqrt{\pi}}[4x(3-4x)^2-2] + C_5\,64\,x(x-1)(1-2x)^2+2\right] \quad (1)$$

Since $y(0)=1$ then

$$C_1\frac{2}{\sqrt{\pi}} + C_2\frac{2}{\sqrt{\pi}}[-2] + C_3\frac{2}{\sqrt{\pi}}[2] + C_4\frac{2}{\sqrt{\pi}}[-2] = 1 \quad (2)$$

and Collocating the equation (1) using the limit points of the following sequence

$$x_i = \left\{\frac{1}{2}\left(1+\cos\left(\frac{(i-1)\pi}{2^{K-1}M}\right)\right)\right\} \text{ where } i = 2,3,4,...\,\&\,K=1, M=5$$

$$\text{When } i=2 \;\Rightarrow x_1=\left\{\frac{1}{2}\left(1+\cos\left(\frac{\pi}{5}\right)\right)\right\}=0.9045085$$

$$i=3 \;\Rightarrow x_2=\left\{\frac{1}{2}\left(1+\cos\left(\frac{2\pi}{5}\right)\right)\right\}=0.6545$$

$$i=4 \;\Rightarrow x_3=\left\{\frac{1}{2}\left(1+\cos\left(\frac{3\pi}{5}\right)\right)\right\}=0.34549$$

$$i=5 \;\Rightarrow x_4=\left\{\frac{1}{2}\left(1+\cos\left(\frac{4\pi}{5}\right)\right)\right\}=0.09549$$

Substituting these Collocating points in the equation (1), we get three more system of algebraic equations say, (3), (4), (5) & (6). Solving these five systems of algebraic equations using MATLAB, we get the values of the coefficients as

$$C_1 = 1.6666$$
$$C_2 = -0.6853$$
$$C_3 = 0.1393$$
$$C_4 = -0.0876$$
$$C_5 = 1.7093e-02$$

Now substituting these values in the equation $y(x)$,we get the solution as

$$y(x) = e^x .$$

Test Problem 4.2 Consider the following multi-pantograph equation [19]

$$y'(x) = -\frac{5}{6}y(x) + 4y(\frac{x}{2}) + 9y(\frac{x}{3}) + x^2 - 1, \quad 0 < x < 1$$

Subject to the initial Condition $y(0) = 1$.

Method of Implementation: Let us assume $y(x) = \sum_{j=0}^{M-1} C_i \psi_j$ for fixed k=1 and M=5

i.e.

$$y(x) = C_1\frac{2}{\sqrt{\pi}} + C_2\frac{2}{\sqrt{\pi}}[4x-2] + C_3\frac{2}{\sqrt{\pi}}[16(x^2-x)+2] + C_4\frac{2}{\sqrt{\pi}}[4x(3-4x)^2-2] + C_5\,64\,x(x-1)(1-2x)^2+2$$

$$\Rightarrow y'(x) = C_2\frac{2}{\sqrt{\pi}}[4] + C_3\frac{2}{\sqrt{\pi}}[16(2x-1)] + C_4\frac{2}{\sqrt{\pi}}[192x^2-192x+36] + C_5\,64\,[16x^3-24x^2+10x-1]$$

Substituting these values of $y(x)$ & $y'(x)$ in the given equation, we get

$$C_2\frac{2}{\sqrt{\pi}}[4] + C_3\frac{2}{\sqrt{\pi}}[16(2x-1)] + C_4\frac{2}{\sqrt{\pi}}[192x^2-192x+36] + C_5\,64\,[16x^3-24x^2+10x-1]$$

=

$$-\frac{5}{6}\left[C_1\frac{2}{\sqrt{\pi}} + C_2\frac{2}{\sqrt{\pi}}[4x-2] + C_3\frac{2}{\sqrt{\pi}}[16(x^2-x)+2] + C_4\frac{2}{\sqrt{\pi}}[4x(3-4x)^2-2] + C_5\,64\,x(x-1)(1-2x)^2+2\right]$$

+

$$4\left[C_1\frac{2}{\sqrt{\pi}} + C_2\frac{2}{\sqrt{\pi}}[4(\frac{x}{2})-2] + C_3\frac{2}{\sqrt{\pi}}[16((\frac{x}{2})^2-(\frac{x}{2}))+2] + C_4\frac{2}{\sqrt{\pi}}[4(\frac{x}{2})(3-4(\frac{x}{2}))^2-2] + C_5\,64\,(\frac{x}{2})((\frac{x}{2})-1)(1-2(\frac{x}{2}))^2+2\right]$$

+

$$9\left[C_1\frac{2}{\sqrt{\pi}}+C_2\frac{2}{\sqrt{\pi}}[4(\frac{x}{3})-2]+C_3\frac{2}{\sqrt{\pi}}[16((\frac{x}{3})^2-(\frac{x}{3}))+2]+C_4\frac{2}{\sqrt{\pi}}[4(\frac{x}{3})(3-4(\frac{x}{3}))^2-2]+C_5\,64\,(\frac{x}{3})((\frac{x}{3})-1)(1-2(\frac{x}{3}))^2+2\right]$$
$$+x^2-1.$$
(1)

Since $y(0)=1$ then

$$C_1\frac{2}{\sqrt{\pi}}+C_2\frac{2}{\sqrt{\pi}}[-2]+C_3\frac{2}{\sqrt{\pi}}[2]+C_4\frac{2}{\sqrt{\pi}}[-2]=1 \qquad (2)$$

and Collocating the equation (1) using the limit points of the following sequence $x_i=\left\{\frac{1}{2}\left(1+\cos\left(\frac{(i-1)\pi}{2^{K-1}M}\right)\right)\right\}$ where $i=2,3,4,...\&\,K=1, M=5$

$$\text{When } i=2 \;\Rightarrow x_1=\left\{\frac{1}{2}\left(1+\cos\left(\frac{\pi}{5}\right)\right)\right\}=0.9045085$$
$$i=3 \;\Rightarrow x_2=\left\{\frac{1}{2}\left(1+\cos\left(\frac{2\pi}{5}\right)\right)\right\}=0.6545$$
$$i=4 \;\Rightarrow x_3=\left\{\frac{1}{2}\left(1+\cos\left(\frac{3\pi}{5}\right)\right)\right\}=0.34549$$
$$i=5 \;\Rightarrow x_4=\left\{\frac{1}{2}\left(1+\cos\left(\frac{4\pi}{5}\right)\right)\right\}=0.09549$$

Substituting these Collocating points in the equation (1), we get three more system of algebraic equations as

$$14.9603C_2-13.7286C_1+37.9583C_3-37.5532C_4+3.9972C_5=-0.5716 \qquad (3)$$
$$19.6618C_2-13.7286C_1+18.3343C_3-37.5532C_4-3.9972C_5=-0.5716 \qquad (4)$$
$$25.4733C_2-13.7286C_1-9.5607C_3-24.3030C_4+46.2155C_5=-0.8806 \qquad (5)$$
$$30.1749C_2-13.7286C_1-35.0720C_3+34.561C_4-20.6078C_5=-0.9909 \qquad (6)$$

Solving these five systems of algebraic equations using MATLAB, we get the values of the coefficients as

$$C_1=16.1636$$
$$C_2=9.5767$$
$$C_3=2.0679$$
$$C_4=0.1299$$
$$C_5=0$$

Now substituting these values in the equation $y(x)$,we get the solution as

$$y(x)=\frac{1675}{72}x^2+\frac{12157}{1296}x^3+\frac{67}{6}x+1$$

Test problem.4.3. Consider the NDDE (Nonlinear DDE)of first order [18] ,

$y'(x)=1-2y^2(\frac{x}{2}),\ 0<x<1$ subjected to the initial condition is y(0)=0, The exact solution is $y(x) = sin(x)$. We solve this equation by the present method with $k = 1$ and different values of M. Table 1 and graph 1 represents the comparison between the absolute error(AE) of approximate solution, analytical solution.

Table 1: Comparison of absolute error (AE) for Example 1 (k=1)

X	Exact solution	AE at M=5	AE at M=6	AE at M=7	AE at M=8	AE at M=9
0.1	0.09983341664	8.0906×10^{-5}	4.8123×10^{-7}	2.6275×10^{-8}	1.7249×10^{-10}	1.8700×10^{-12}
0.2	0.19866933079	8.8860×10^{-5}	2.7973×10^{-8}	7.0278×10^{-10}	1.6474×10^{-10}	2.6637×10^{-11}
0.3	0.29552020666	3.2727×10^{-5}	2.6251×10^{-9}	2.5380×10^{-8}	6.4387×10^{-10}	2.7476×10^{-11}
0.4	0.38941834230	1.5278×10^{-5}	3.4758×10^{-7}	4.1051×10^{-8}	3.0730×10^{-10}	1.8539×10^{-12}
0.5	0.47942553860	2.2769×10^{-5}	5.5569×10^{-7}	1.9710×10^{-8}	9.3084×10^{-11}	1.2950×10^{-11}
0.6	0.56464247339	1.2190×10^{-5}	3.4597×10^{-7}	3.0745×10^{-9}	2.2579×10^{-10}	2.7166×10^{-11}
0.7	0.64421768723	7.0072×10^{-5}	4.9143×10^{-8}	8.0286×10^{-9}	4.9983×10^{-10}	2.5429×10^{-12}
0.8	0.71735609089	1.1839×10^{-4}	7.4765×10^{-8}	2.7793×10^{-8}	1.0529×10^{-10}	1.8333×10^{-12}
0.9	0.78332690962	1.1965×10^{-4}	3.9225×10^{-7}	2.8050×10^{-9}	1.2919×10^{-10}	2.2902×10^{-11}

Fig 1: Physical interpretation of AE for Example 1 for different values of M.

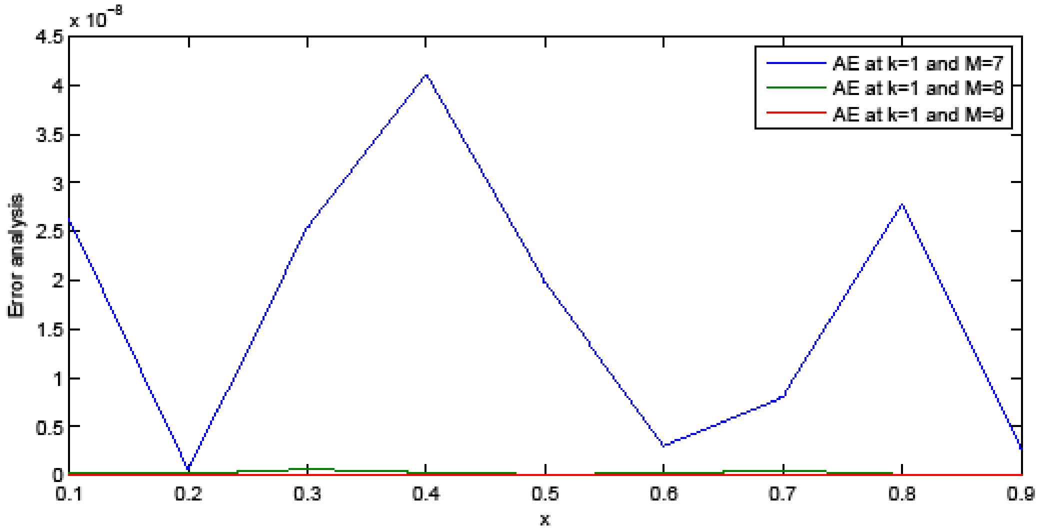
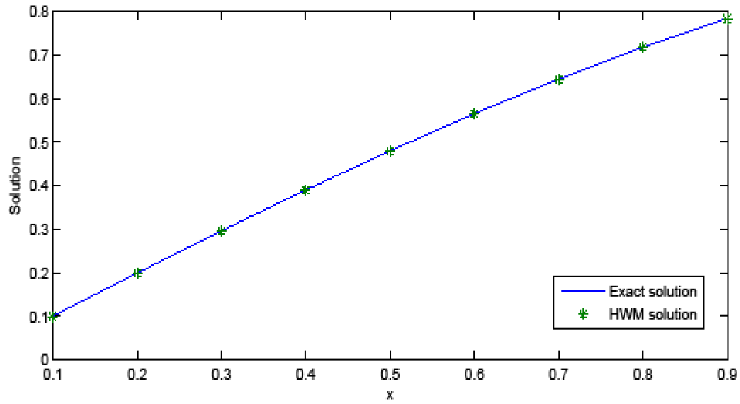


Fig 2: Graphical interpretation of Exact solution with approximate solution at k=1 and M=9 for Example 1.



Test Problem.4.4. Consider the following DDE equation

$$y'(x) - y(\frac{x}{2}) = 0, \quad 0 < x \le 1$$

Subject to the initial Condition $y(0) = 1$.

With the exact solution $y(x) = \sum_{k=0}^{\infty} \frac{\frac{1}{2}^{\frac{1}{2}k(k-1)}}{k!} x^k$ if we solve above Equation using the algorithm described in the section 3 for the case corresponding to *k=1* and *M=5*. After performing some manipulations, the components of the vector C are obtained. They are as follows;

$$C_1 = 2.2617,\ C_2 = -1.904,\ C_3 = 1.1775,\ C_4 = -2.6561, C_5 = 17680$$

Then corresponding solution which is same as exact, $y(x) = \sum_{k=0}^{\infty} \frac{\frac{1}{2}^{\frac{1}{2}k(k-1)}}{k!} x^k$

5. Conclusion

In this work, we developed the Hermite wavelet based numerical Method for DDE's. First, we proposed the method and then we applied to linear and nonlinear DDE. Therefore, it is seen that the method is very effective to solve differential equations with bounded and unbounded delays. Same procedure can be extended for higher order also with slight modification in the proposed method.

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